Noncommutative Field Theory from Quantum Mechanical Space-Space Noncommutativity

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ABSTRACT

We investigate the incorporation of space noncommutativity into field theory by extending to the spectral continuum the minisuperspace action of the quantum mechanical harmonic oscillator propagator with an enlarged Heisenberg algebra. In addition to the usual $\star$-product deformation of the algebra of field functions, we show that the parameter of noncommutativity can occur in noncommutative field theory even in the case of free fields without self-interacting potentials.

1. Introduction

Particle Quantum Mechanics can be viewed in the free field or weak coupling limit as a minisuperspace sector of quantum field theory where most of the degrees of freedom have been frozen. It is thus a very convenient arena for further investigating the implications of the quantum mechanical spacetime noncommutativity in the formulation of field theories, as well as for evaluating the justification of some statements that are considered as generally accepted wisdom among practitioners of noncommutative field theory. *Cf e.g.* [1] -[14] and references therein for related works, albeit in a somewhat different spirit from the problem considered here, on noncommutative quantum mechanics. Further, in [15] noncommutativity was considered within the context of the Weyl-Wigner-Gröenewold-Moyal (WWGM) formalism for extended Heisenberg algebras, and its relation to the Bopp shift map (or what some authors refer to as the quantum mechanical equivalent of the Seiberg-Witten map) for expressing the algebra of extended Heisenberg operators in terms of their commutative counterparts was discussed and results were compared for problems previously studied in some of the above cited works. Moreover, the canonical, noncanonical and the possible quantum mechanical nonunitarity nature of some of these maps was additionally analyzed in [16, 17]. The results found there are conceptually relevant to our approach here, since as shown in several of the examples considered, transforming a problem in NCQM into a commutative one does not always lead to two unitarily equivalent quantum mechanical formulations.

A case in point, arises when we compare some of our results and their physical implications with those obtained by the procedure followed in Ref [18]. The comparison is quite pertinent since both approaches are analogous in that they both have a quantum mechanical minisuperspace as a starting point for a construction of a field theoretical model. Indeed, the original quantum Hamiltonian in [18], modulo some irrelevant normalizations, is the same as the one considered here. On the other hand, the extended original Heisenberg algebra used in that work (Eq(2.6)) is different from ours because the authors there require to introduce (for their latter arguments) also noncommutativity of the momenta operators. Making then a linear transformation (actually a Bopp shift map) to a new set of quantum variables which satisfies the usual Heisenberg algebra results in their new Hamiltonian (2.9). The remainder of the construction in [18] follows from the above. Note, however, that the two decoupled quantum oscillators obtained in that work are not the same as ours (to see this it suffices to set $\hat{B} = 0$, in their equations (2.10) and compare them with our equation (3.60)). Thus the quantum mechanical problems implied by Eqs. (2.5) and (2.9) in [18] are not unitarily equivalent. In fact, the quantum mechanical problem that is actually considered there is that of a two dimensional anharmonic oscillator with a particular choice of frequencies containing some constant terms labeled with the symbols $\hat{\theta}$ and $\hat{B}$, which can not truly be identified with the noncommutativity of any of the observables generated by the Heisenberg algebra (2.8) characterizing the quantum problem that at the end of the day is involved in that work.
Moreover, the field constructed in [18] is a complex scalar field, which is not so in our case, and the Feynman propagator derived there and given in Eq(3.5) is quite different from ours (*cf* Eq(4.66)). The most important difference being that (3.5) in Ref [18] satisfies a highly non-local differential equation which violates both ordinary as well as twisted Poincaré invariance, while the symmetries of the Feynman propagator we derive in this work are in agreement with recent results on twisted NCQFT.

Based on the above remarks and recalling that observables in quantum mechanics are represented by Hermitian operators acting on a Hilbert space, noncommutativity of the dynamical variables of a quantum mechanical system can be readily understood as the noncommutativity of their corresponding operators. In this way the physical argument that measurements below distances of the order of the

Planck length loose operational significance [19], can be mathematically described by extending the usual Heisenberg algebra of ordinary quantum mechanics to one including the noncommutativity of the operators related to the spacetime dynamical variables. Consistently in this paper we shall therefore use a quantum basis which is fully compatible with the noncommutativity of the coordinates.

In particular in order to formulate space noncommutativity in Quantum Mechanics we use the extended Heisenberg algebra with generators satisfying the commutation relations

$$
\begin{aligned}
\left[\hat{Q}_i, \hat{Q}_j\right] &= i\theta_{ij}, \\
\left[\hat{Q}_i, \hat{P}_j\right] &= i\hbar\delta_{ij}, \\
\left[\hat{P}_i, \hat{P}_j\right] &= 0,
\end{aligned}
$$

(1.1)

(these could of course be generalized even more by also postulating noncommutativity of the momenta). The parameters θ_{ij} of noncommutativity in (1.1) have dimensions of $(length)^2$ and can, in general, be themselves arbitrary antisymmetric functions of the spacetime operators. However, most of the work so far appearing in the literature assumes for simplicity that these parameters are constant, and so shall we in what follows. The observables formed from the generators in (1.1) act on a Hilbert space which is assumed to be the same as the one for ordinary quantum mechanics, for any of the admissible realizations of the extended noncommutative Heisenberg algebra.

Furthermore, utilizing the WWGM formalism for a quantum mechanical system, with observables obeying the above extended Heisenberg algebra, we showed in a previous paper [17] that the Weyl equivalent to a Heisenberg operator $\Omega(\hat{\mathbf{P}}, \hat{\mathbf{Q}}, t)$ satisfies the differential equation

$$
(1.2) \qquad \frac{\partial \Omega_W(\mathbf{p}, \mathbf{q}, t)}{\partial t} = -\frac{2}{\hbar} H_W \sin\left[\frac{1}{2}\left(\hbar\Lambda + \sum_{i \neq j} \theta_{ij}\Lambda'_{ij}\right)\right] \Omega_W(\mathbf{p}, \mathbf{q}, t),
$$

where

$$
(1.3) \qquad
\begin{aligned}
\Lambda &= \overleftarrow{\nabla}_{\mathbf{q}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{q}}, \\
\Lambda'_{ij} &= \overleftarrow{\partial}_{q_i} \overrightarrow{\partial}_{q_j},
\end{aligned}
$$

and H_W is the Weyl equivalent to the quantum Hamiltonian.
Making use of (1.2) we further showed in [17] that in the WWGM formalism of quantum mechanics, the labeling variables $\mathbf{q}, \mathbf{p}$, can be interpreted as canonical classical dynamical variables provided their algebra $\mathcal{A}$ is modified with a multiplication given by the star-product:

$$
(1.4) \qquad q_i \star_\theta q_j := q_i \left(e^{\frac{1}{2}\sum_{k,l}\theta_{kl}\overleftarrow{\partial}_{q_k}\overrightarrow{\partial}_{q_l}}\right) q_j.
$$

Applying these results to the simple case of a two dimensional harmonic oscillator satisfying the algebra (1.1), which is taken as the unfrozen mode, or the one particle sector of a two-component vector (or composite system) field, and using spectral analysis in order to reconstruct the corresponding quantum field, we shall show how the parameter of the quantum mechanical noncommutativity appears in the theory even for the case of a free field. This novel result, which as we shall see is a quite natural consequence of our approach, and contrasts with the usually made assumption that the presence of noncommutativity in field theory is manifested only through the deformation of the multiplication in the algebra of the fields [20], [21], [22].

2. THE QUANTUM MECHANICS OF THE HARMONIC OSCILLATOR IN NONCOMMUTATIVE SPACE

As discussed in [17], a configuration space basis for the quantum mechanics with an extended Heisenberg algebra generated by (1.1) is not an admissible basis, since the position operators $\hat{Q}_i, \hat{Q}_j, \quad i \neq j$, do not simultaneously form part of a complete set of commuting observables. For $i, j = 1, 2$, then the only admissible bases for such a case are either one of the 3 sets of kets $\{|q_1, p_2\rangle\}$, $\{|q_2, p_1\rangle\}$ and $\{|p_1, p_2\rangle\}$, where the labels of the kets are the eigenvalues of the possible sets of commuting observables.

Let us now consider the first of these bases and use the WWGM formalism and the results in [17] in order to evaluate the transition amplitude $\langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle$, for a quantum 2-dimensional harmonic oscillator with Hamiltonian

$$(2.5) \qquad \hat{H} = \frac{1}{2m}\left(\hat{P}_1^2 + \hat{P}_2^2\right) + \frac{m\omega^2}{2}\left(\hat{Q}_1^2 + \hat{Q}_2^2\right).$$

¿From the results in Sec. 2 of the above cited paper, it can be seen that this transition amplitude is given by

$$
\begin{aligned}
\langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle &= \langle q_1''(t_1), p_2''(t_1)|e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}|q_1'(t_1), p_2'(t_1)\rangle \\
(2.6) \qquad &= \text{Tr}[\rho e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}] \\
&= \int dp_1 dp_2 dq_1 dq_2 \rho_W \, e^{\frac{\theta}{\hbar}p_1\partial_{q_2}} e_\star^{-\frac{i}{\hbar}H_W(t_2-t_1)},
\end{aligned}
$$

where $\theta := \theta_{12}$,

$$(2.7) \qquad \rho := |q_1'(t_1), p_2'(t_1)\rangle\langle q_1''(t_1), p_2''(t_1)|,$$

ρ_W is its corresponding Weyl function:

$$(2.8) \qquad \rho_W = (2\pi\hbar)^{-2}\int d\xi d\eta e^{-\frac{i}{\hbar}(\eta q_2 - \xi p_1)}\langle q_1 - \frac{\xi}{2}, p_2 - \frac{\eta}{2}|\rho|q_1 + \frac{\xi}{2}, p_2 + \frac{\eta}{2}\rangle,$$

and $H_W := \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{m\omega^2}{2}\left(q_1^2 + q_2^2\right)$ is the Weyl function associated with the quantum Hamiltonian (2.5). Substituting (2.7) into (2.8) gives

$$(2.9) \qquad \rho_W = \frac{4}{(2\pi\hbar)^2}\delta(q_1'' + q_1' - 2q_1)\delta(p_2'' + p_2' - 2p_2)\exp\left[-\frac{i}{\hbar}(2p_2 - 2p_2')q_2 + \frac{i}{\hbar}(2q_1 - 2q_1')p_1\right],$$

which, when inserted in its turn into (2.6), yields

$$
\begin{aligned}
\langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle &= \int dp_1 dq_2 e^{-\frac{i}{\hbar}(p_2''-p_2')q_2}\exp\{\frac{i}{\hbar}[(q_1'' - q_1') + \frac{\theta}{\hbar}(p_2'' - p_2')]p_1\} \\
(2.10) \qquad &\times \; (e_\star^{-\frac{i}{\hbar}H_W(t_2-t_1)})(p_1, \frac{p_2' + p_2''}{2}, \frac{q_1' + q_1''}{2}, q_2).
\end{aligned}
$$

Note now that for an infinitesimal transition with $t_1 = t$, $t_2 = t + \delta t$ and $q_1'' - q_1' = \dot{q}_1''\delta t$, $p_2'' - p_2' = \dot{p}_2''\delta t$, (2.10) reads

$$(2.11) \qquad \langle q_1''(t + \delta t), p_2''(t + \delta t)|q_1'(t), p_2'(t)\rangle = e^{\frac{i}{\hbar}[\dot{q}_1'p_1 - \dot{p}_2'q_2 + \frac{\theta}{\hbar}\dot{p}_2'p_1]\delta t}e^{-\frac{i}{\hbar}H_{cl}(p_1, p_2', q_1', q_2)\delta t},$$

where H_{cl} $(= H_W$ for the case here considered) is the classical Hamiltonian resulting from making the replacements $\hat{Q} \to q$ and $\hat{P} \to p$ in the original quantum Hamiltonian (2.5). Following Feynman's path integral formalism, the transition over a finite time interval is then given by

$$(2.12) \qquad \langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle \sim \int \mathcal{D}q_1\mathcal{D}p_2\mathcal{D}p_1\mathcal{D}q_2\exp\{\frac{i}{\hbar}\int_{t_1}^{t_2}[\dot{q}_1 p_1 - \dot{p}_2 q_2 + \frac{\theta}{\hbar}\dot{p}_2 p_1 - H_{cl}]dt\}.$$

This result (for an alternate derivation see [23] and related work in [24]-[27]) provides an univocal procedure for obtaining the Feynman propagator in spacetime noncommutative quantum mechanics as well as the expression for the deformed classical action, which in our particular case is given by

$$(2.13) \qquad S(q_1, p_2, q_2, p_1, t) = \int_{t_1}^{t_2} [\dot{q}_1 p_1 - \dot{p}_2 q_2 + \frac{\theta}{\hbar}\dot{p}_2 p_1 - H_{cl}]dt.$$

Let us next re-write the action (2.13) in the form

$$(2.14) \qquad S = \int dt \left[p_1\dot{q}_1 - \dot{p}_2(q_2 - \frac{\theta}{\hbar}p_1) - \frac{p_1^2}{2m} - \frac{p_2^2}{2m} - \frac{m\omega^2}{2}q_1^2 - \frac{m\omega^2}{2}q_2^2 \right],$$

which, when setting

$$(2.15) \qquad \tilde{q}_2 = q_2 - \frac{\theta}{\hbar}p_1,$$

results in

$$(2.16) \qquad S = \int dt \left[p_1\dot{q}_1 - \dot{p}_2\tilde{q}_2 - \frac{p_1^2}{2m} - \frac{p_2^2}{2m} - \frac{m\omega^2}{2}q_1^2 - \frac{m\omega^2}{2}(\tilde{q}_2 + \frac{\theta}{\hbar}p_1)^2 \right].$$

Fixing now p_1 and $\tilde{q}_2$ at the end points and varying with respect to these variables we get

$$(2.17) \qquad \dot{q}_1 = \frac{p_1}{m} + \frac{m^2\omega^2\theta}{\hbar}(\tilde{q}_2 + \frac{\theta}{\hbar}p_1),$$

$$(2.18) \qquad \dot{p}_2 = -m\omega^2(\tilde{q}_2 + \frac{\theta}{\hbar}p_1).$$

¿From the above we derive

$$(2.19) \qquad p_1 = m\dot{q}_1 + \frac{m\theta}{\hbar}\dot{p}_2$$

$$(2.20) \qquad \tilde{q}_2 = -\frac{1}{m\omega^2}[\dot{p}_2 + \frac{m^2\omega^2\theta}{\hbar}(\dot{q}_1 + \frac{\theta}{\hbar}\dot{p}_2)].$$

Substituting these last expressions into (2.16) shows that (2.12) may be reduced to

$$(2.21) \qquad \langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle \sim \int\int \mathcal{D}q_1 \mathcal{D}p_2 e^{\frac{i}{\hbar}S(q_1, p_2, t)},$$

with

$$(2.22) \qquad S(q_1, p_2, t) = \int dt \left[\frac{m}{2}\dot{q}_1^2 + \frac{m\theta}{\hbar}\dot{p}_2\dot{q}_1 + \left(\frac{1}{2m\omega^2} + \frac{m\theta^2}{2\hbar^2}\right)\dot{p}_2^2 - \frac{p_2^2}{2m} - \frac{m\omega^2}{2}q_1^2 \right].$$

Note that by varying (2.22), it follows that the canonical dynamical variables q_1 and p_2 obey the set of second order coupled ordinary differential equations

$$(2.23) \qquad \begin{pmatrix} \ddot{q}_1 \\ \ddot{p}_2 \end{pmatrix} = - \begin{pmatrix} \frac{m^2\omega^4\theta^2}{\hbar^2} + \omega^2 & -\frac{\omega^2\theta}{\hbar} \\ -\frac{m^2\omega^4\theta}{\hbar} & \omega^2 \end{pmatrix} \begin{pmatrix} q_1 \\ p_2 \end{pmatrix},$$

which, when diagonalized, decouple into two harmonic oscillators with frequencies given by

$$(2.24) \qquad \omega_{1,2} = \omega \left[1 + \frac{m^2\omega^2\theta^2}{2\hbar^2} \pm \frac{m\omega\theta}{2\hbar}\sqrt{4 + \frac{m^2\omega^2\theta^2}{\hbar^2}} \right]^{\frac{1}{2}}.$$

Hence, the energy eigenvalues of (2.5) are

$$(2.25) \qquad E = \hbar\omega_1(n_1 + \frac{1}{2}) + \hbar\omega_2(n_2 + \frac{1}{2}).$$

It is pertinent to emphasize here that the change of variables at the classical level involved in Eq. (2.15) does not correspond to a Bopp shift, so it also does not follow that making such a change of

variables in the action (2.14) implies that we are passing from NCQM to ordinary quantum mechanics.

2.1. **Hamiltonian formulation.** Consider now the Lagrangian L in the action (2.22) and make the identifications

(2.26)
$$z_1 := q_1, \quad z_2 := \frac{p_2}{m\omega},$$

so that both z_1 and z_2 have dimension of length. Furthermore, introducing the dimensionless quantity $\tilde{\theta}$:

(2.27)
$$\tilde{\theta} = \frac{m\omega\theta}{\hbar},$$

with m, ω, being some characteristic mass and frequency, respectively, to be further specified below, we can then write

(2.28)
$$L = \frac{1}{2}\left[\dot{z}_1^2 - \omega^2 z_1^2 + \dot{z}_2^2 - \omega^2 z_2^2 + 2\tilde{\theta}\dot{z}_1\dot{z}_2 + \tilde{\theta}^2\dot{z}_2^2\right].$$

The momenta canonical to the z_i's are

(2.29)
$$\begin{aligned}
\pi_1 &= \dot{z}_1 + \tilde{\theta}\dot{z}_2, \\
\pi_2 &= \dot{z}_2 + \tilde{\theta}\dot{z}_1 + \tilde{\theta}^2\dot{z}_2.
\end{aligned}$$

Inverting (2.29) we have

(2.30)
$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1+\tilde{\theta}^2 & -\tilde{\theta} \\ -\tilde{\theta} & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix},$$

from where it follows that

(2.31)
$$H = \pi_1\dot{z}_1 + \pi_2\dot{z}_2 - L = \frac{1}{2}\left[(1+\tilde{\theta}^2)\pi_1^2 + \pi_2^2 - 2\tilde{\theta}\pi_1\pi_2 + \omega^2 z_1^2 + \omega^2 z_2^2\right].$$

Making use of the theory of quadrics we can diagonalize (2.31) by first solving for the eigenvalues $\lambda_{1,2}$ of the characteristic determinant of the matrix

$$\begin{pmatrix} \frac{1}{2}(1+\tilde{\theta}^2) & -\frac{\tilde{\theta}}{2} \\ -\frac{\tilde{\theta}}{2} & \frac{1}{2} \end{pmatrix}.$$

We thus get

(2.32)
$$\lambda_{1,2} = \frac{1}{2}\left(1 + \frac{\tilde{\theta}^2}{2} \pm \frac{\tilde{\theta}}{2}\sqrt{4+\tilde{\theta}^2}\right).$$

Hence

(2.33)
$$\begin{aligned}
H &= (\pi_1, \pi_2)\left(\tilde{M}\right)(M)\begin{pmatrix} \frac{1}{2}(1+\tilde{\theta}^2) & -\frac{\tilde{\theta}}{2} \\ -\frac{\tilde{\theta}}{2} & \frac{1}{2} \end{pmatrix}\left(\tilde{M}\right)(M)\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} + \frac{\omega^2}{2}\left(z_1^2 + z_2^2\right) \\
&= (\pi_1', \pi_2')\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \pi_1' \\ \pi_2' \end{pmatrix} + \frac{\omega^2}{2}\left(z_1'^2 + z_2'^2\right),
\end{aligned}$$

where

(2.34)
$$\begin{pmatrix} \pi_1' \\ \pi_2' \end{pmatrix} = (M)\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \quad \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = (M)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

and the entries of the symmetric matrix $(M) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ are given by

(2.35)
$$m_{11} = -\frac{1}{\sqrt{1 + \frac{\omega_2^2}{\omega^2}}}, \quad m_{12} = -\frac{\frac{\omega_2^2}{\omega^2} - 1}{\tilde{\theta}\sqrt{1 + \frac{\omega_2^2}{\omega^2}}},$$

(2.36)
$$m_{21} = \frac{1}{\sqrt{1 + \frac{\omega_1^2}{\omega^2}}}, \quad m_{22} = \frac{\frac{\omega_1^2}{\omega^2} - 1}{\tilde{\theta}\sqrt{1 + \frac{\omega_1^2}{\omega^2}}}$$

where, by using (2.24) and (2.27), one can readily verify that $m_{12} = m_{21}$ as required.
If we finally let

(2.37)
$$z_i' = (\lambda_i)^{\frac{1}{2}} x_i, \quad \pi_i' = (\lambda_i)^{-\frac{1}{2}} \pi_{x_i}, \quad i = 1, 2,$$

we arrive at

(2.38)
$$H = \pi_{x_1}^2 + \pi_{x_2}^2 + \frac{1}{4}\left(\omega_1^2 x_1^2 + \omega_2^2 x_2^2\right).$$

It should be clear from the above calculations that the transformed variables x_i, π_{x_i} remain canonically conjugate to each other. Thus it follows from the Hamilton equations that

(2.39)
$$\pi_{x_i} = \frac{1}{2}\dot{x}_i,$$

so the Lagrangian (2.28) now reads

(2.40)
$$L = \frac{1}{4}\left(\dot{x}_1^2 + \dot{x}_2^2 - \omega_1^2 x_1^2 - \omega_2^2 x_2^2\right).$$

Variation of this expression with respect to x_i yields

(2.41)
$$\ddot{x}_i + \omega_i^2 x_i = 0,$$

which are indeed the equations of motion for two decoupled harmonic oscillators with respective frequencies ω_i, as asserted previously.
Furthermore, it can be readily verified that the point transformations

(2.42)
$$\begin{pmatrix} \pi_{x_1} \\ \pi_{x_2} \end{pmatrix} = \begin{pmatrix} m_{11}\sqrt{\lambda_1} & m_{12}\sqrt{\lambda_1} \\ m_{21}\sqrt{\lambda_2} & m_{22}\sqrt{\lambda_2} \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix},$$

and

(2.43)
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{m_{11}}{\sqrt{\lambda_1}} & \frac{m_{12}}{\sqrt{\lambda_2}} \\ \frac{m_{21}}{\sqrt{\lambda_1}} & \frac{m_{22}}{\sqrt{\lambda_2}} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

are canonical, with generating function

(2.44)
$$F_2(z_1, z_2, \pi_{x_1}, \pi_{x_2}) = \sum_{i,j} \frac{m_{ij}}{\sqrt{\lambda_i}} z_j \pi_{x_i}.$$

We also have that when substituting (2.43) into the Lagrangian (2.40) we recover (2.28) and that the Jacobian of each the transformations (2.42) and (2.43) is equal to $\frac{1}{2}$, so that

$$\mathcal{D}x_1 \mathcal{D}x_2 = \frac{1}{2}\mathcal{D}z_1 \mathcal{D}z_2.$$

Consequently, the quantum mechanics derived from the path integral with the action (2.22) is unitarily equivalent to the path integral formulation based on the action resulting from the diagonalized Lagrangian (2.40).

3. Field theoretical model

Paralleling standard quantum field theory we next construct a noncommutative field theory over a $(1 + 2)$-Minkowski space by taking an infinite superposition of the quantum mechanical harmonic oscillator minisuperspaces described by (2.40). Each of these oscillators consists of the pair $x_1(\mathbf{k}), x_2(\mathbf{k})$, labeled by the continuous parameter $\mathbf{k}$ and satisfying (2.41). Thus in our construction, the quantum mechanical spacial noncommutativity will reflect itself both in the deformation parameter dependence of the different frequencies of the pairs of oscillators, as well as in the twisting of the product of the algebra of the resulting fields. Consequently this simple model shows that spacetime noncommutativity can be present in field theory even in the absence of self-interaction potentials.

Let us consider a field system $\Phi_i(\mathbf{q}, t)$, $i = 1, 2$, over a $(1 + 2)$-Minkowski space-time, satisfying the uncoupled Klein-Gordon field equations

$$(3.45) \qquad \begin{pmatrix} \Box^2 + \mu_1^2 & 0 \\ 0 & \Box^2 + \mu_2^2 \end{pmatrix} \begin{pmatrix} \Phi_1(q_1, q_2, t) \\ \Phi_2(q_1, q_2, t) \end{pmatrix} = 0,$$

where

$$(3.46) \qquad \Phi_i(\mathbf{q}, t) = (2\pi)^{-1} \int d\mathbf{k} \; x_i(\mathbf{k}, t) \; e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}},$$

and

$$(3.47) \qquad e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}} := 1 + i\mathbf{k}.\mathbf{q} + \frac{1}{2}(i\mathbf{k}.\mathbf{q}) \star_\theta (i\mathbf{k}.\mathbf{q}) + \dots.$$

Note that in the above definition of the field system in terms of its Fourier transform we have used the star-exponential for describing plane waves. Our rationale for this is based on the observation made in ([15]) where, by making use of the WWGM formalism and elements of quantum group theory, we show that quantum noncommutativity of coordinate operators in the extended Heisenberg algebra leads to a deformed product of the classical dynamical variables that is inherited to the level of quantum field theory. This deformed product is the so called Moyal star-product defined in (1.4). Thus, expressing the fields as in (3.46) guarantees explicitly that they are elements of the deformed algebra $\mathcal{A}_\theta$ with the $\star$-multiplication.

Note also that in (3.45) the D'Alembertian is given by

$$(3.48) \qquad \Box^2 = \partial_t^2 - \bar{\partial}_i^\dagger \bar{\partial}_i,$$

with the anti-hermitian derivation $\bar{\partial}_i$ defined by [28]:

$$(3.49) \qquad \bar{\partial}_i = \theta_{ij}^{-1} \, ad_{q_j}, \quad \text{and} \quad \bar{\partial}_i^\dagger = -\bar{\partial}_i, \quad i = 1, 2,$$

and where the adjoint action is realized by the twisted product commutator

$$(3.50) \qquad [q_i, q_j]_{\star_\theta} := q_i \star_\theta q_j - q_j \star_\theta q_i.$$

Thus, the algebra (1.4) has been incorporated into (3.45) through the defining Fourier transformation equation (3.46) for the fields since these, as functions of the q_i's, they inherit the $\star$-multiplication and are therefore also elements of the twisted algebra $\mathcal{A}_\theta$.

Now, by making use of the Baker-Campbell-Hausdorff theorem, together with the commutator (3.50) as well as of the identity $[q_2, q_1^n]_{\star_\theta} = -in\theta q_1^{(n-1)}$, we have that

$$\begin{aligned} \bar{\partial}_1(e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}}) & = \theta^{-1}[q_2, e^{ik_1 q_1} e^{ik_2 q_2} e^{\frac{i}{2}k_1 k_2 \theta}]_{\star_\theta} \\ (3.51) \qquad & = k_1 e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}}, \end{aligned}$$

and (recalling that $\bar{\partial}_i^\dagger = -\bar{\partial}_i$)

(3.52)
$$\bar{\partial}_1^\dagger \bar{\partial}_1 (e_{\star_\theta}^{i\mathbf{k}\cdot\mathbf{q}}) = -k_1^2 e_{\star_\theta}^{i\mathbf{k}\cdot\mathbf{q}}.$$

Similarly

(3.53)
$$\bar{\partial}_2^\dagger \bar{\partial}_2 (e_{\star_\theta}^{i\mathbf{k}\cdot\mathbf{q}}) = -k_2^2 e_{\star_\theta}^{i\mathbf{k}\cdot\mathbf{q}}.$$

We therefore find that the field equations (3.45) read

(3.54)
$$(\Box^2 + \mu_i^2)\Phi_i(\mathbf{q}, t) = (2\pi)^{-1} \int d\mathbf{k} [\ddot{x}_i + (\mathbf{k}^2 + \mu_i^2)x_i] e_{\star_\theta}^{i\mathbf{k}\cdot\mathbf{q}} = 0, \quad i = 1, 2.$$

Using next the orthonormality

(3.55)
$$(2\pi)^{-2} \int \int dq_1 dq_2 e_{\star_\theta}^{i\mathbf{k}\cdot\mathbf{q}} e_{\star_\theta}^{-i\mathbf{k}'\cdot\mathbf{q}} = \delta(\mathbf{k} - \mathbf{k}'),$$

and the dispersion relation

(3.56)
$$\mathbf{k}^2 + \mu_i^2 = k_0^2 = \omega_i^2(\mathbf{k}),$$

we obtain from the right hand of (3.54):

(3.57)
$$\ddot{x}_i(\mathbf{k}, t) + \omega_i^2(\mathbf{k})\mathbf{x_i}(\mathbf{k}, \mathbf{t}) = \mathbf{0}.$$

Observe that $\omega_i(\mathbf{k})$, $i = 1, 2$, in (3.56) is given by (2.24) with $\omega \to \omega(\mathbf{k})$ and $\theta \to \theta(\mathbf{k})$ being now respectively the wave vector dependent frequency in (1.1) and the noncommutative parameter of the quantum mechanical system for each $\mathbf{k}$ in the spectral decomposition (3.46). Comparing (3.57) with (2.41), and observing that according to our definition (2.27) we now have $\theta(\mathbf{k}) = \frac{\hbar\tilde{\theta}}{m\omega(\mathbf{k})}$, we choose $\theta(\mathbf{k})$ such that $\tilde{\theta}$ remains a pure number independent of $\mathbf{k}$. We then have that the Lagrangian (2.40) for the pair of decoupled harmonic oscillators $x_i(\mathbf{k}, t)$ can be seen, for a fixed value of the continuum parameter $\mathbf{k}$, as a minisuperspace of the full field theory characterized by the action:

(3.58)
$$S = \int dt dq_1 dq_2\, \mathcal{L} = \frac{1}{2} \int dt dq_1 dq_2 \left[\dot{\Phi}_1^\dagger \star_\theta \dot{\Phi}_1 - (\bar{\partial}_i\Phi_1)^\dagger \star_\theta \bar{\partial}_i\Phi_1 - \mu_1^2 \Phi_1^\dagger \star_\theta \Phi_1 \right.$$
$$+ \dot{\Phi}_2^\dagger \star_\theta \dot{\Phi}_2 - (\bar{\partial}_i\Phi_2)^\dagger \star_\theta \bar{\partial}_i\Phi_2 - \mu_2^2 \Phi_2^\dagger \star_\theta \Phi_2 + \frac{1}{2}(\Phi_1^\dagger \star_\theta J_1(\mathbf{q}, t)$$
$$\left. + J_1^\dagger(\mathbf{q}, t) \star_\theta \Phi_1) + \frac{1}{2}(\Phi_2^\dagger \star_\theta J_2(\mathbf{q}, t) + J_2^\dagger(\mathbf{q}, t) \star_\theta \Phi_2) \right],$$

after adding two arbitrary external driving sources.

Note that in the above expression we have formally included the $\star$-product for the algebra of the fields, even though in fact, in the absence of field interaction potentials, these could be ignored in view of the identity

(3.59)
$$\int dq_1 dq_2 f(\mathbf{q}) \star_\theta g(\mathbf{q}) = \int dq_1 dq_2 f(\mathbf{q}) g(\mathbf{q}),$$

which follows directly by parts integration. However, also note that the noncommutativity parameter $\tilde{\theta}$ will still be present in the frequencies $\omega_i(\mathbf{k})$ even in such a case, since these now read

(3.60)
$$\omega_{1,2}(\mathbf{k}) = \omega(\mathbf{k}) \left[1 + \frac{\tilde{\theta}^2}{2} \pm \frac{\tilde{\theta}}{2}\sqrt{4 + \tilde{\theta}^2} \right]^{\frac{1}{2}}.$$

4. Path integral and Feynman propagator

In order to derive the Feynman propagator for our theory, we use (3.46) and a similar expression for the Fourier transform $\tilde{F}_i$ of the sources J_i together with (3.55), as well as the transformations

$$
\begin{aligned}
x_i((\mathbf{k}, t) &= (2\pi)^{-\frac{1}{2}} \int dk_0 e^{ik_0 t} \tilde{x}_i(\mathbf{k}, k_0), \\
F_i((\mathbf{k}, t) &= (2\pi)^{-\frac{1}{2}} \int dk_0 e^{ik_0 t} \tilde{F}_i(\mathbf{k}, k_0).
\end{aligned}
$$

(4.61)

We thus get

$$
\begin{aligned}
S = \frac{1}{2} \int dk_0 d\mathbf{k} \Bigg[&(k_0^2 - \mathbf{k}^2 - \mu^2) \left(\sum_{i=1,2} \tilde{x}_i(\mathbf{k}, k_0) \tilde{x}_i(\mathbf{k}, -k_0) \right) \\
&+ \tilde{x}_1(\mathbf{k}, k_0) \tilde{F}_1(\mathbf{k}, -k_0) + \tilde{x}_1(\mathbf{k}, -k_0) \tilde{F}_1(\mathbf{k}, k_0) \\
&+ \tilde{x}_2(\mathbf{k}, k_0) \tilde{F}_2(\mathbf{k}, -k_0) + \tilde{x}_2(\mathbf{k}, -k_0) \tilde{F}_2(\mathbf{k}, k_0) \Bigg].
\end{aligned}
$$

(4.62)

Following standard procedures (see *e.g.* [29]), we now make the change of variables

$$
\begin{aligned}
\tilde{x}_1(\mathbf{k}, k_0) &= Z_1(\mathbf{k}, k_0) + \beta(k_0)\tilde{F}_1(\mathbf{k}, k_0) + \gamma(k_0)F_2(\mathbf{k}, k_0), \\
\tilde{x}_2(\mathbf{k}, k_0) &= Z_2(\mathbf{k}, k_0) + \lambda(k_0)\tilde{F}_1(\mathbf{k}, k_0) + \nu(k_0)F_2(\mathbf{k}, k_0).
\end{aligned}
$$

(4.63)

Inserting (4.63) into (4.62) and requiring that terms linear in the Z_i's cancel, allows us to fix the parameters $\beta, \gamma, \lambda, \nu$ as:

$$
\begin{aligned}
\beta(k_0) &= (k_0^2 - \mathbf{k}^2 - \mu^2)^{-1}, \\
\lambda(k_0) &= \gamma(k_0) = 0, \\
\nu(k_0) &= -(k_0^2 - \mathbf{k}^2 - \mu^2)^{-1}.
\end{aligned}
$$

(4.64)

If we next replace (4.64) into the action resulting from (4.62) by the above procedure, we derive the following contribution to the integrand in that action from the terms quadratic in the sources:

$$
\begin{aligned}
\langle Z_0[J] \rangle := -\frac{1}{2} \int \ldots \int d\mathbf{q}\, d\mathbf{q}'\, dt\, dt' \left(J_1^\dagger(\mathbf{q}, t) \quad J_2^\dagger(\mathbf{q}, t) \right) \\
\times \begin{pmatrix} D_1(\mathbf{q} - \mathbf{q}', \mathbf{t} - \mathbf{t}') & 0 \\ 0 & D_2(\mathbf{q} - \mathbf{q}', \mathbf{t} - \mathbf{t}') \end{pmatrix} \begin{pmatrix} J_1(\mathbf{q}', t') \\ J_2(\mathbf{q}', t') \end{pmatrix},
\end{aligned}
$$

(4.65)

where $D_i(\mathbf{q} - \mathbf{q}', t - t')$ are the Feynman propagators:

$$
D_i(\mathbf{q} - \mathbf{q}', t - t') = (2\pi)^{-3} \int \ldots \int d\mathbf{k}\, dk_0 \left(\frac{e^{-i[k_0(t-t') - \mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')]}}{k_0^2 - \omega_i^2(\mathbf{k}) + i\epsilon} \right), \quad i = 1, 2,
$$

(4.66)

and the $\omega_i^2(\mathbf{k})$ are given by (3.60).

Note that these propagators satisfy the Klein-Gordon equations

$$
\left(\Box^2 + \mu_i^2 \right) D_i(\mathbf{q} - \mathbf{q}', t - t') = -\delta(\mathbf{q} - \mathbf{q}')\delta(t - t').
$$

(4.67)

Observe also that (4.67) is invariant under the twisted Poincaré transformations discussed in [30], since the D'Alembertian, as defined in (3.48), is invariant under these transformations and the indices $i = 1, 2$, are not space-time indices.

In consequence of the above, the vacuum to vacuum amplitude for our theory is thus given by

$$
W[J] = W[0] e^{\frac{i}{\hbar} \langle Z_0[J] \rangle},
$$

(4.68)

and the classical fields $\Phi^{(0)}_{(cl)i} \equiv -i\frac{\delta \ln W_0}{\delta J_i^\dagger(\mathbf{q},t)} = \frac{\delta Z_0}{\delta J_i^\dagger(\mathbf{q},t)}$ satisfy the driven Klein-Gordon field equations

$$(4.69) \qquad\qquad (\Box^2 + \mu_i^2)\Phi^{(0)}_{(cl)i} = \frac{1}{2}J_i.$$

5. Second Quantization

Let us promote the x_i in (3.46) to the rank of operators and, similarly to that equation, let us define field canonical momenta by

$$(5.70) \qquad\qquad \hat{\Pi}_i = (2\pi)^{-1}\int d\mathbf{k}\; \hat{\pi}_i(\mathbf{k},t)\, e^{-i\mathbf{k}.\mathbf{q}}_{\star_\theta},$$

with $\hat{x}_i, \hat{\pi}_j$, satisfying now the commutation relations

$$(5.71) \qquad \begin{aligned} [\hat{x}_i(\mathbf{k},t), \hat{\pi}_j(\mathbf{k}',t)] &= i\hbar\delta_{ij}\delta(\mathbf{k}-\mathbf{k}'), \\ [\hat{x}_i(\mathbf{k},t), \hat{x}_j(\mathbf{k}',t)] &= 0, \\ [\hat{\pi}_i(\mathbf{k},t), \hat{\pi}_j(\mathbf{k}',t)] &= 0. \end{aligned}$$

Assuming further that $\hat{\Phi}_i$ and $\hat{\Pi}_i$ are real, we have by Hermicity that

$$(5.72) \qquad\qquad \hat{x}_i^\dagger(\mathbf{k},t) = \hat{x}_i(-\mathbf{k},t), \quad \hat{\pi}_i^\dagger(\mathbf{k},t) = \hat{\pi}_i(-\mathbf{k},t),$$

and we also take $\omega_i(\mathbf{k}) = \omega_i(-\mathbf{k})$.
Next let

$$(5.73) \qquad \begin{aligned} \hat{x}_i(\mathbf{k},t) &= \sqrt{\frac{\hbar}{2\omega_i(\mathbf{k})}}\left(\hat{a}_i(\mathbf{k},t) + \hat{a}_i^\dagger(-\mathbf{k},t)\right), \\ \hat{\pi}_i(\mathbf{k},t) &= i\sqrt{\frac{\hbar\omega_i(\mathbf{k})}{2}}\left(\hat{a}_i^\dagger(\mathbf{k},t) - \hat{a}_i(-\mathbf{k},t)\right). \end{aligned}$$

It readily follows from (5.73) and (5.71) that

$$(5.74) \qquad \begin{aligned} [\hat{a}_i(\mathbf{k},t), \hat{a}_j^\dagger(\mathbf{k}',t)] &= \delta_{ij}\delta(\mathbf{k}-\mathbf{k}'), \\ [\hat{a}_i(\mathbf{k},t), \hat{a}_j(\mathbf{k}',t)] &= 0, \\ [\hat{a}_i^\dagger(\mathbf{k},t), \hat{a}_j^\dagger(\mathbf{k}',t)] &= 0. \end{aligned}$$

So $\hat{a}_i^\dagger$ and $\hat{a}_i$, are the usual Fock creation and destruction operators. Note however that while particle-antiparticle degeneracy at the dispersion relation level is preserved for a given value of the label $i = 1, 2$, the energies of the particles-antiparticles created (destroyed) by $\hat{a}_i^\dagger$ ($\hat{a}_i$) are different and are given by $\hbar\omega_i$.

6. Discussion and Conclusions

Spacetime noncommutativity in field theory is understood in some circles as a merely convenient way to describe a special type of interaction. Such a description consisting in mathematically deforming the product in the algebra of field functions by means of the so-called Moyal star-product. However, as we tried to stress throughout the paper, referring to the formalism under such premises as spacetime noncommutativity is, at best, a misnomer since the arguments of the fields are parameters of the theory. Speaking about noncommutativity in this context then has little physical basis, beyond the rather loose analogy of the Moyal product with the Groenewold- Moyal product occurring in the WWGM phase-space formulation of quantum mechanics. One of our contentions here has been, however, that there is more physical substance to that designation if one recalls the operational nature of observables in

quantum mechanics from where noncommutativity of the dynamical variables of the system is readily understood then as the noncommutatitivity of their corresponding operators. Furthermore, based on the concept that quantum mechanics can be viewed as a minisuperspace sector of field theory, where only a few degrees of freedom are unfrozen, we have used the quantum mechanics of a harmonic oscillator over an extended Heisenberg algebra, to construct a field theoretical model which inherits the space-space noncommutativity of the quantum mechanical problem.

An interesting feature of our construction is that it shows that the global symmetry of the original theory (2.5) is broken by the noncommutativity. This in turn implies that if at the level of field theory the index tagging the fields denotes a composite system of scalar fields (and not the components of a vector field), then the noncommutativity can be seen as giving rise to a field doublet (or more generally an n-tuplet) of slightly different masses where classical Lorentz symmetry for each member is broken, but each one satisfies a deformed Klein-Gordon equation which is invariant under a twisted Lorentz symmetry. On the other hand, if the labeling of the fields is taken as corresponding to that of a vector field of spacetime dimensions then, because of the mass differences, both classical and twisted Lorentz invariance are broken by the noncommutativity.

This symmetry breaking and mass differences resulting from the presence of noncommutativity is in some way reminiscent of the spontaneous symmetry breaking mechanism that occurs in the Standard Model, but without the appearance of a Goldstone boson.

In addition, by thinking of noncommutativity of spacetime as the quantum mechanical operator algebra expressing the loss of operational meaning for localization at distances of orders smaller than the Planck length, it then follows that minisuperspaces based on noncommutative spacetimes have to be at least of two dimensions, and the fields constructed from them must necessarily contain the presence of the parameter of noncommutativity even in the absence of self-interacting potentials.

An alternate way to mathematically express the physical argument that measurements below distances of the order of the Planck length loose operational significance, can be accomplished, both at the quantum mechanical and field theoretical level, by using parametrization invariance of the action and following the canonical quantization approach of embedding a spatial manifold Σ in the spacetime manifold. Such an approach, whereby the embedding variables acquire a dynamical interpretation, which, in turn, gives physical sense to their noncommutativity and is achieved by the inclusion of a general symplectic structure in the formalism, has been analyzed extensively by the authors elsewhere [31]. The deformed algebra of the constraints resulting from the parametrization and general symplectic structure of the theory is particularly convenient for analyzing the twisting of its symmetries and for indeed thinking of a true physical spacetime noncommutativity as underlying the merely axiomatic mathematical deformation of the algebra product describing a certain type of interactions in field theory.

Finally, we note that although our construction has been restricted for simplicity to two spacial dimensions and to bosonic fields, it can be generalized to allow for higher dimensional spaces in a conceptually straightforward (albeit algebraically more complicated) way, and to the case of fermionic fields by including Grassmanian variables in the construction of the spectral oscillators.

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